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EQUATIONS OF PROJECTILE MOTION IN A FLEXIBLE GUN TUBE

Interim Report
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This interim report presents the formulation of a system of simultaneous differential equations which describes the general motion of a projectile of finite geometry and inertia traveling in a flexible gun tube. The formulation permits the projectile six degrees-of-freedom relative to the gun tube; three orthogonal translational motions of the projectile c.g. relative to the instantaneous gun tube axis and three (Eulerian) rotational motions. cont

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cont → of the projectile about its c.g. The formulation is presented in terms of interior ballistics data, projectile design data, and gun tube design and motion data. The formulation accounts for projectile spin, mass eccentricity, elastic deformation of the projectile rotating band and bourrelet, and projectile/bore interfacial friction and torque transmission. Furthermore, the formulation is unrestricted regarding the nature of gun tube motion.

The formulation has been compared with projectile descriptions and formulations employed by other investigators and, based upon this comparison, is considered to be the most generally applicable projectile in-bore motion formulation appearing in recent literature.

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1. INTRODUCTION

The following interim technical report has been prepared by S&D Dynamics, Inc. under Contract No. DAAG29-83-C-0004 to the U.S. Army Research Office, Research Triangle Park, NC.

This interim report presents the detailed formulation of the equations of motion of a projectile of finite geometric and inertial properties traveling in a flexible gun tube. The formulation permits the projectile six degrees-of-freedom relative to the gun tube; three orthogonal translational motions of the projectile c.g. relative to the instantaneous gun tube axis and three (Eulerian) rotations of the projectile about its c.g. (related to projectile pitch, yaw and roll motions). The formulation accounts for projectile spin, mass eccentricity, elastic deformation of the projectile rotating band and bourrelet, and projectile/bore interfacial friction and torque transmission. Furthermore, the formulation is unrestricted regarding the nature of gun tube motion.

The detailed formulation is presented in Section 2. A comparison of this formulation with other projectile descriptions and formulations is presented in Section 3. Conclusions are presented in Section 4. References cited are presented in Section 5. The Euler angles and associated transformations employed are defined in the Appendix.

2. FORMULATION OF PROJECTILE EQUATIONS OF MOTION

The simultaneous system of differential equations which describe the general, six degree-of-freedom motion of a projectile traveling in a flexible gun tube are formulated in this section. The pertinent equations are formulated in terms of the loads and moments applied to the projectile in Section 2.1. Simplification of the projectile angular velocity and acceleration expressions is discussed in Section 2.2. The applied loads and moments acting on the projectile are defined in Section 2.3. The solution technique, which allows simultaneous solution of the equations herein developed with the equations of the gun dynamics simulation code previously developed, is discussed briefly in Section 2.4 and will be presented in greater detail in a separate report.

2.1 Projectile Equations of Motion in Terms of Applied Loads and Moments

The projectile is described subject to the following assumptions:

(i) the projectile is assumed to consist of a main body, rotating band and bourrelet;

(ii) the projectile main-body is assumed to behave as a rigid body of finite geometry and inertia;

(iii) the rotating band and bourrelet are assumed to behave elastically (although not necessarily linearly elastic);

(iv) the projectile c.g. is assumed to be eccentrically located relative to its geometric center;

(v) the projectile is permitted six degrees-of-freedom relative to the gun tube, namely pitch, yaw and roll about Eulerian axes, and translational motions along body-fixed (orthogonal) axes.

The following coordinate systems are introduced for the purpose of formulating the equations of motion of the projectile:

(i) S', with coordinates (x',y',z') and unit triad $(\hat{i}',\hat{j}',\hat{k}')$, is defined as an inertial reference frame fixed at the initial projectile position such that its orientation coincides with the global (earth-fixed) coordinate system defined in the gun dynamics simulation code;

(ii) S_o, with coordinates (x_o,y_o,z_o) and unit triad $(\hat{i}_o,\hat{j}_o,\hat{k}_o)$, is defined as an intermediate reference frame whose origin translates with the projectile along the gun tube axis and rotates with the projectile about the instantaneous tangent to the gun tube axis. $\hat{i}_o$ is directed toward the gun tube muzzle, along the instantaneous gun tube axis. At the initial projectile position, $\hat{j}_o$ is directed along the radius from the gun tube centerline to the projectile c.g.;

(iii) S, with coordinates (x,y,z) and unit triad $(\hat{i},\hat{j},\hat{k})$, is defined as a body-fixed coordinate system whose origin is located at the projectile c.g. $\hat{i}$ is directed toward the projectile nose, parallel to its geometric axis. At the initial projectile position, $\hat{j}$ is parallel to $\hat{j}_o$.

It is noted that the kinematic relations between S_o and S' incorporate gun tube motion, as well as the two degrees-of-freedom of the projectile corresponding to translational motion along the gun tube axis and spin (both of which are prescribed by interior ballistics data). The remaining four degrees-of-freedom of the projectile are formulated in S relative to S_o. Referring to Figure 1, these are defined as the translational displacements of the projectile c.g. in the $\hat{j}_o$-$\hat{k}_o$ plane (denoted as y_{cg} and z_{cg}), and the Euler angles corresponding to projectile pitch and yaw, namely ψ and θ respectively.

For the purpose of formulating the velocity and acceleration of the origin of S_o relative to S', the Euler angles $(\psi_o, \theta_o, \varphi_o)$ defined in the Appendix are introduced subject to the transformation

$$\left.\begin{aligned} \hat{i}' &= \ell^{o}_{11}\hat{i}_o + \ell^{o}_{12}\hat{j}_o + \ell^{o}_{13}\hat{k}_o \\ \hat{j}' &= \ell^{o}_{21}\hat{i}_o + \ell^{o}_{22}\hat{j}_o + \ell^{o}_{23}\hat{k}_o \\ \hat{k}' &= \ell^{o}_{31}\hat{i}_o + \ell_{32}\hat{j}_o + \ell^{o}_{33}\hat{k}_o \end{aligned}\right\} \tag{1}$$

where (referring to the Appendix),

$$\left.\begin{aligned} \ell^{o}_{11} &= \cos\psi_o \cos\theta_o \\ \ell^{o}_{12} &= \cos\psi_o \sin\theta_o \sin\varphi_o - \sin\psi_o \cos\varphi_o \\ \ell^{o}_{13} &= \cos\psi_o \sin\theta_o \cos\varphi_o + \sin\psi_o \sin\varphi_o \\ \ell^{o}_{21} &= \sin\psi_o \cos\theta_o \\ \ell^{o}_{22} &= \sin\psi_o \sin\theta_o \sin\varphi_o + \cos\psi_o \cos\varphi_o \\ \ell^{o}_{23} &= \sin\psi_o \sin\theta_o \cos\varphi_o - \cos\psi_o \sin\varphi_o \\ \ell^{o}_{31} &= -\sin\theta_o \\ \ell^{o}_{32} &= \cos\theta_o \sin\varphi_o \\ \ell^{o}_{33} &= \cos\theta_o \cos\varphi_o \end{aligned}\right\} \tag{2}$$

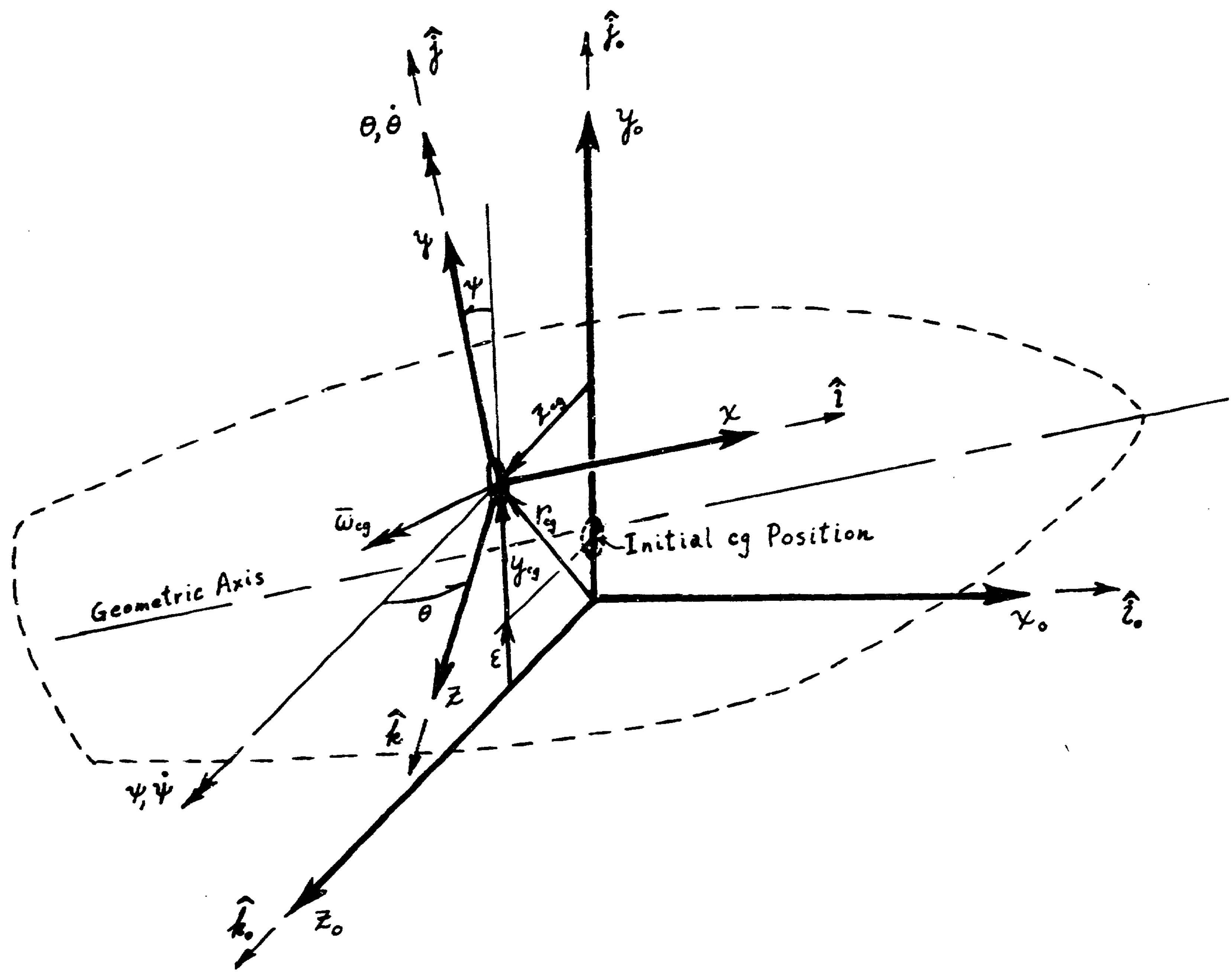

Figure 1 - <u>Orientation of S Relative to S_o</u>

Since S_o is constrained to move with the projectile, the position vector of its origin, $\bar{R}_o$, as shown in Figure 2, has components which are functions of time, t, as well as distance along the gun tube axis, s. Hence, the velocity of the origin of S_o relative to S', namely $\bar{v}_o$ (= $d\bar{R}_o/dt$), is given as

$$\bar{v}_o = \left[\frac{\partial x'}{\partial t} + \frac{ds}{dt}\frac{\partial x'}{\partial s}\right]\hat{i}' + \left[\frac{\partial y'}{\partial t} + \frac{ds}{dt}\frac{\partial y'}{\partial s}\right]\hat{j}' + \left[\frac{\partial z'}{\partial t} + \frac{ds}{dt}\frac{\partial z'}{\partial s}\right]\hat{k}' \tag{3}$$

Since $\hat{i}_o$ lies along the instantaneous tangent to the gun tube axis it follows that

$$\frac{\partial x'}{\partial s} = \ell_{11}^{\circ}; \quad \frac{\partial y'}{\partial s} = \ell_{21}^{\circ}; \quad \frac{\partial z'}{\partial s} = \ell_{31}^{\circ} \tag{4}$$

Hence,

$$\bar{v}_o = \left[\frac{\partial x'}{\partial t} + \ell_{11}^{\circ}\frac{ds}{dt}\right]\hat{i}' + \left[\frac{\partial y'}{\partial t} + \ell_{21}^{\circ}\frac{ds}{dt}\right]\hat{j}' + \left[\frac{\partial z'}{\partial t} + \ell_{31}^{\circ}\frac{ds}{dt}\right]\hat{k}' \tag{5}$$

Applying the transformation presented in equation (1), and noting that ds/dt denotes the instantaneous projectile velocity, v_p (as prescribed by interior ballistics data), there results from equation (5) the velocity of S_o in the form

$$\begin{aligned}\bar{v}_o = \left[v_p + \ell_{11}^{\circ}\frac{\partial x'}{\partial t} + \ell_{21}^{\circ}\frac{\partial y'}{\partial t} + \ell_{31}^{\circ}\frac{\partial z'}{\partial t}\right]\hat{i}_o + \left[\ell_{12}^{\circ}\frac{\partial x'}{\partial t} + \ell_{22}^{\circ}\frac{\partial y'}{\partial t} + \ell_{32}^{\circ}\frac{\partial z'}{\partial t}\right]\hat{j}_o + \\ + \left[\ell_{13}^{\circ}\frac{\partial x'}{\partial t} + \ell_{23}^{\circ}\frac{\partial y'}{\partial t} + \ell_{33}^{\circ}\frac{\partial z'}{\partial t}\right]\hat{k}_o\end{aligned} \tag{6}$$

Since S' is earth-fixed, the instantaneous acceleration of S_o is readily obtained by differentiating equation (3) with respect to time noting that

$$\frac{d}{dt}(\ ) = \frac{\partial(\ )}{\partial t} + v_p\frac{\partial(\ )}{\partial s} \tag{7}$$

from which there results

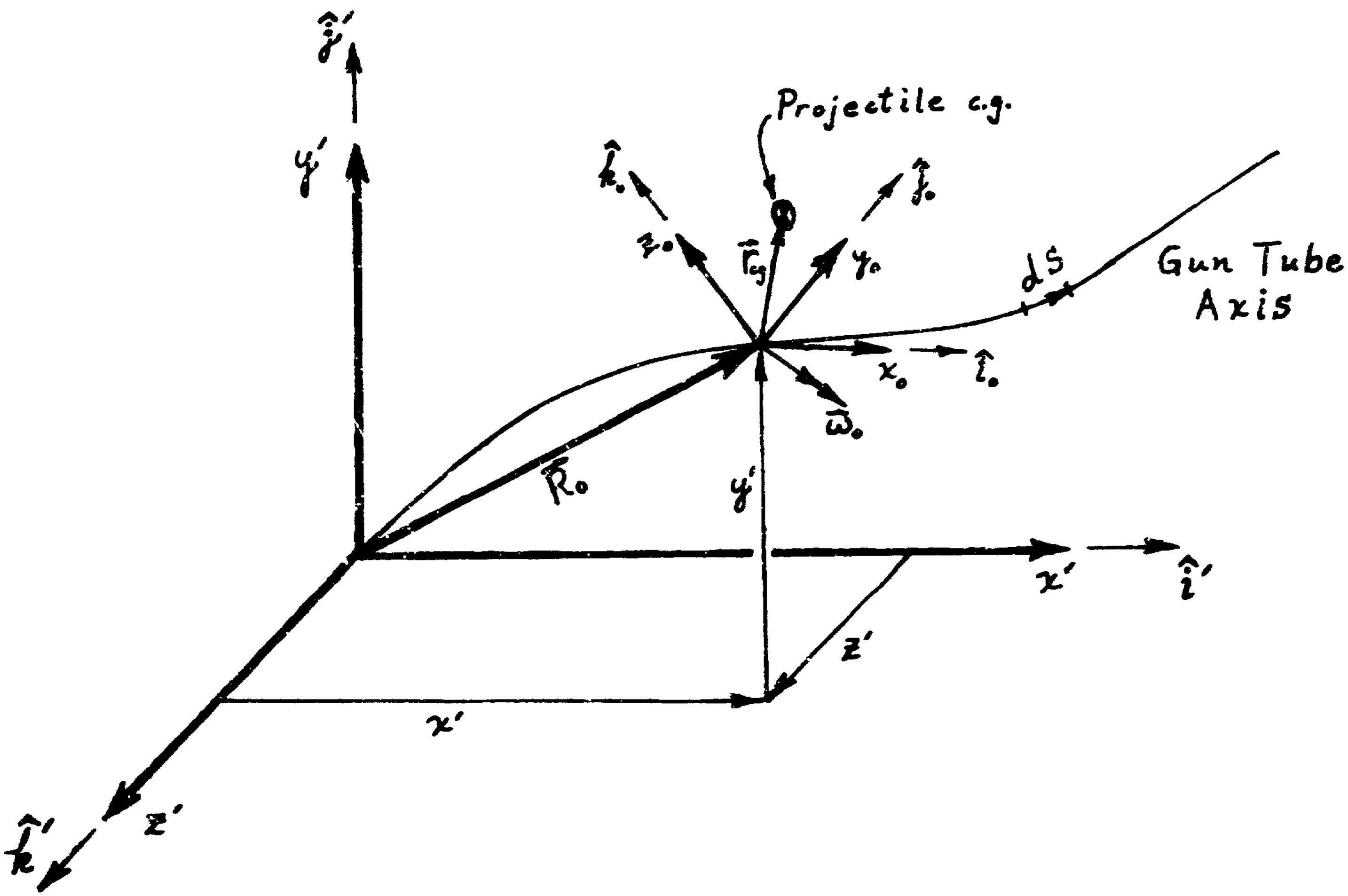

Figure 2 - Orientation of S_o Relative to S'

$$\bar{a}_o = \left[\frac{\partial^2 x'}{\partial t^2} + 2v_p \frac{\partial^2 x'}{\partial s \partial t} + v_p^2 \frac{\partial^2 x'}{\partial s^2} + a_p \frac{\partial x'}{\partial s} \right] \hat{i}' +$$
$$+ \left[\frac{\partial^2 y'}{\partial t^2} + 2v_p \frac{\partial^2 y'}{\partial s \partial t} + v_p^2 \frac{\partial^2 y'}{\partial s^2} + a_p \frac{\partial y'}{\partial s} \right] \hat{j}' +$$
$$+ \left[\frac{\partial^2 z'}{\partial t^2} + 2v_p \frac{\partial^2 z'}{\partial s \partial t} + v_p^2 \frac{\partial^2 z'}{\partial s^2} + a_p \frac{\partial z'}{\partial s} \right] \hat{k}' \tag{8}$$

where a_p denotes dv_p/dt.

Once again applying the transformation presented in equation (1), and noting equation (4), there results the acceleration of S_o in the form

$$\bar{a}_o = a_{x_o} \hat{i}_o + a_{y_o} \hat{j}_o + a_{z_o} \hat{k}_o \tag{9}$$

where,

$$\left.\begin{aligned}
a_{x_o} &= a_p + \ell^o_{11} \frac{\partial^2 x'}{\partial t^2} + \ell^o_{21} \frac{\partial^2 y'}{\partial t^2} + \ell^o_{31} \frac{\partial^2 z'}{\partial t^2} \\
a_{y_o} &= \ell^o_{12} \frac{\partial^2 x'}{\partial t^2} + \ell^o_{22} \frac{\partial^2 y'}{\partial t^2} + \ell^o_{32} \frac{\partial^2 z'}{\partial t^2} + 2v_p \left[\frac{\partial \psi_o}{\partial t} \cos\theta_o \cos\varphi_o - \frac{\partial \theta_o}{\partial t} \sin\varphi_o \right] + \\
&\quad + v_p^2 \left[\frac{\partial \psi_o}{\partial s} \cos\theta_o \cos\varphi_o - \frac{\partial \theta_o}{\partial s} \sin\varphi_o \right] \\
a_{z_o} &= \ell^o_{13} \frac{\partial^2 x'}{\partial t^2} + \ell^o_{23} \frac{\partial^2 y'}{\partial t^2} + \ell^o_{33} \frac{\partial^2 z'}{\partial t^2} - 2v_p \left[\frac{\partial \psi_o}{\partial t} \cos\theta_o \sin\varphi_o + \frac{\partial \theta_o}{\partial t} \cos\varphi_o \right] + \\
&\quad - v_p^2 \left[\frac{\partial \psi_o}{\partial s} \cos\theta_o \sin\varphi_o + \frac{\partial \theta_o}{\partial s} \cos\varphi_o \right]
\end{aligned}\right\} \tag{10}$$

Noting that S_o is constrained to move with the gun tube in the $\hat{j}_o$-$\hat{k}_o$ plane, it has angular velocity, $\bar{\omega}_o$, relative to S'. The expression for $\bar{\omega}_o$ (written relative to S_o), is given as

$$\bar{\omega}_o = \omega_{x_o} \hat{i}_o + \omega_{y_o} \hat{j}_o + \omega_{z_o} \hat{k}_o \tag{11}$$

where (referring to the Appendix),

$$\left.\begin{aligned} \omega_{x_o} &= \dot{\varphi}_o - \dot{\psi}_o \sin\theta_o \\ \omega_{y_o} &= \dot{\theta}_o \cos\varphi_o + \dot{\psi}_o \cos\theta_o \sin\varphi_o \\ \omega_{z_o} &= -\dot{\theta}_o \sin\varphi_o + \dot{\psi}_o \cos\theta_o \cos\varphi_o \end{aligned}\right\} \tag{12}$$

and $\dot{\psi}_o$, $\dot{\theta}_o$ and $\dot{\varphi}_o$ are obtained applying equation (7).

Noting further that

$$\frac{d(\bar{\ })}{dt} = (\dot{\bar{\ }}) + \bar{\omega} \times (\bar{\ }) \tag{13}$$

there results the angular acceleration of S_o relative to S'

$$\dot{\bar{\omega}}_o = \dot{\omega}_{x_o} \hat{i}_o + \dot{\omega}_{y_o} \hat{j}_o + \dot{\omega}_{z_o} \hat{k}_o \tag{14}$$

where (from equation (12)),

$$\left.\begin{aligned} \dot{\omega}_{x_o} &= \ddot{\varphi}_o - \ddot{\psi}_o \sin\theta_o - \dot{\psi}_o \dot{\theta}_o \cos\theta_o \\ \dot{\omega}_{y_o} &= \ddot{\theta}_o \cos\varphi_o + \ddot{\psi}_o \cos\theta_o \sin\varphi_o - \dot{\psi}_o \dot{\theta}_o \sin\theta_o \sin\varphi_o + \omega_{z_o} \dot{\varphi}_o \\ \dot{\omega}_{z_o} &= -\ddot{\theta}_o \sin\varphi_o + \ddot{\psi}_o \cos\theta_o \cos\varphi_o - \dot{\psi}_o \dot{\theta}_o \sin\theta_o \cos\varphi_o - \omega_{y_o} \dot{\varphi}_o \end{aligned}\right\} \tag{15}$$

Equations (6), (9), (11) and (14) prescribe the linear and angular velocities and accelerations of S_o relative to S'. It remains to prescribe the motion of the projectile c.g. relative to S_o.

Letting $\bar{v}_{cg}$ and $\bar{a}_{cg}$ respectively denote the translational velocity and acceleration of S relative to S_o, it follows from

Figure 1 that

$$\left.\begin{aligned} \bar{v}_{cg} &= \dot{y}_{cg}\hat{j}_o + \dot{z}_{cg}\hat{k}_o \\ \bar{a}_{cg} &= \ddot{y}_{cg}\hat{j}_o + \ddot{z}_{cg}\hat{k}_o \end{aligned}\right\} \qquad (16)$$

Letting $\bar{\omega}_{cg}$ denote the angular velocity of S relative S_o, and further specifying that

$$\bar{\omega}_{cg} = \omega_{x_{cg}}\hat{i} + \omega_{y_{cg}}\hat{j} + \omega_{z_{cg}}\hat{k} \qquad (17)$$

the components of $\bar{\omega}_{cg}$ are prescribed (analogous to equation (12)) as functions of the Euler angles (ψ, θ, φ) of S relative to S_o in the form

$$\left.\begin{aligned} \omega_{x_{cg}} &= \dot{\varphi} - \dot{\psi}\sin\theta \\ \omega_{y_{cg}} &= \dot{\theta}\cos\varphi + \dot{\psi}\cos\theta\sin\varphi \\ \omega_{z_{cg}} &= -\dot{\theta}\sin\varphi + \dot{\psi}\cos\theta\cos\varphi \end{aligned}\right\} \qquad (18)$$

Noting that S_o is prescribed to rotate with S, equation (17) must be subjected to the constraint

$$\bar{\omega}_{cg} \cdot \hat{i}_o = 0 \qquad (19)$$

Introducing the transformation from S to S_o, namely

$$\left.\begin{aligned} \hat{i}_o &= \ell_{11}\hat{i} + \ell_{12}\hat{j} + \ell_{13}\hat{k} \\ \hat{j}_o &= \ell_{21}\hat{i} + \ell_{22}\hat{j} + \ell_{23}\hat{k} \\ \hat{k}_o &= \ell_{31}\hat{i} + \ell_{32}\hat{j} + \ell_{33}\hat{k} \end{aligned}\right\} \qquad (20)$$

and noting that the direction cosines, ℓ_{ij}, are defined in terms of the Euler angles (ψ,θ,φ) analogous to the definition presented in equation (2), there results from equations (17) thru (20)

$$\dot{\varphi} = \left(\frac{\tan\psi}{\cos\theta}\right)\dot{\theta} \tag{21}$$

which prescribes the dependence between the Euler angles of S required to satisfy the constraint presented in equation (19).

Equations (16) and (17) (subject to equation (21)) respectively prescribe the translational and angular motions of the projectile c.g. relative to S_o.

Referring to Figures 1 and 2, the instantaneous position vector of S relative to S' is given by $\bar{R}_o+\bar{r}_{cg}$. Hence, noting equation (13), the (absolute) translational acceleration of the projectile c.g. (relative to S'), namely $\bar{a}$, is given by

$$\bar{a} = \bar{a}_o + \bar{a}_{cg} + 2\,\bar{\omega}_o \times \bar{v}_{cg} + \dot{\bar{\omega}}_o \times \bar{r}_{cg} + \bar{\omega}_o \times (\bar{\omega}_o \times \bar{r}_{cg}) \tag{22}$$

while the (total) angular velocity of the projectile (relative to S') is given by

$$\bar{\omega} = \bar{\omega}_o + \bar{\omega}_{cg} \tag{23}$$

Applying equation (13) to each vector on the right-hand side of equation (23) there results the (total) angular acceleration of the projectile (relative to S')

$$\dot{\bar{\omega}} = \dot{\bar{\omega}}_o + \dot{\bar{\omega}}_{cg} + \bar{\omega}_o \times \bar{\omega}_{cg} \tag{24}$$

Substituting equations (9), (11), (14), and (16) into equation (22) there results

$$\bar{a} = \left[a_{x_0} + 2\omega_{y_0}\dot{z}_{cg} - 2\omega_{z_0}\dot{y}_{cg} + z_{cg}(\dot{\omega}_{y_0} + \omega_{x_0}\omega_{z_0}) - (y_{cg}+\varepsilon)(\dot{\omega}_{z_0} - \omega_{x_0}\omega_{y_0})\right]\hat{i}_0 +$$

$$+ \left[a_{y_0} + \ddot{y}_{cg} - 2\omega_{x_0}\dot{z}_{cg} - z_{cg}(\dot{\omega}_{x_0} - \omega_{z_0}\omega_{y_0}) - (y_{cg}+\varepsilon)(\omega_{x_0}^2 + \omega_{z_0}^2)\right]\hat{j}_0 +$$

$$+ \left[a_{z_0} + \ddot{z}_{cg} + 2\omega_{x_0}\dot{y}_{cg} + (y_{cg}+\varepsilon)(\dot{\omega}_{x_0} + \omega_{z_0}\omega_{y_0}) - z_{cg}(\omega_{x_0}^2 + \omega_{y_0}^2)\right]\hat{k}_0 \tag{25}$$

Transforming equation (11) from S_o to S via equation (20) and substituting the result, along with equation (17), into equation (23), there results

$$\bar{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k} \tag{26}$$

where,

$$\left.\begin{aligned} \omega_x &= \omega_{x_{cg}} + \ell_{11}\omega_{x_0} + \ell_{21}\omega_{y_0} + \ell_{31}\omega_{z_0} \\ \omega_y &= \omega_{y_{cg}} + \ell_{12}\omega_{x_0} + \ell_{22}\omega_{y_0} + \ell_{32}\omega_{z_0} \\ \omega_z &= \omega_{z_{cg}} + \ell_{13}\omega_{x_0} + \ell_{23}\omega_{y_0} + \ell_{33}\omega_{z_0} \end{aligned}\right\} \tag{27}$$

Transforming equation (14) from S_o to S via equation (20), differentiating the components of equation (17) with respect to time, and substituting these results, along with equation (17), into equation (24), there results

$$\dot{\bar{\omega}} = \dot{\omega}_x\hat{i} + \dot{\omega}_y\hat{j} + \dot{\omega}_z\hat{k} \tag{28}$$

where,

$$
\left.
\begin{aligned}
\dot{\omega}_x &= \dot{\omega}_{x_{cg}} + \ell_{11}\dot{\omega}_{x_0} + \ell_{21}\dot{\omega}_{y_0} + \ell_{31}\dot{\omega}_{z_0} + \omega_{y_{cg}}(\ell_{13}\omega_{x_0} + \ell_{23}\omega_{y_0} + \ell_{33}\omega_{z_0}) + \\
&\qquad - \omega_{z_{cg}}(\ell_{12}\omega_{x_0} + \ell_{22}\omega_{y_0} + \ell_{32}\omega_{z_0}) \\
\dot{\omega}_y &= \dot{\omega}_{y_{cg}} + \ell_{12}\dot{\omega}_{x_0} + \ell_{22}\dot{\omega}_{y_0} + \ell_{32}\dot{\omega}_{z_0} + \omega_{x_{cg}}(\ell_{13}\omega_{x_0} + \ell_{23}\omega_{y_0} + \ell_{33}\omega_{z_0}) + \\
&\qquad - \omega_{z_{cg}}(\ell_{11}\omega_{x_0} + \ell_{21}\omega_{y_0} + \ell_{31}\omega_{z_0}) \\
\dot{\omega}_z &= \dot{\omega}_{z_{cg}} + \ell_{13}\dot{\omega}_{x_0} + \ell_{23}\dot{\omega}_{y_0} + \ell_{33}\dot{\omega}_{z_0} + \omega_{y_{cg}}(\ell_{11}\omega_{x_0} + \ell_{21}\omega_{y_0} + \ell_{31}\omega_{z_0}) + \\
&\qquad - \omega_{x_{cg}}(\ell_{12}\omega_{x_0} + \ell_{22}\omega_{y_0} + \ell_{32}\omega_{z_0})
\end{aligned}
\right\} \qquad (29)
$$

and (from equation (18)),

$$
\left.
\begin{aligned}
\dot{\omega}_{x_{cg}} &= \ddot{\varphi} - \ddot{\psi}\sin\theta - \dot{\psi}\dot{\theta}\cos\theta \\
\dot{\omega}_{y_{cg}} &= \ddot{\theta}\cos\varphi + \ddot{\psi}\cos\theta\sin\varphi - \dot{\psi}\dot{\theta}\sin\theta\sin\varphi + \omega_{z_{cg}}\dot{\varphi} \\
\dot{\omega}_{z_{cg}} &= -\ddot{\theta}\sin\varphi + \ddot{\psi}\cos\theta\cos\varphi - \dot{\psi}\dot{\theta}\sin\theta\cos\varphi - \omega_{y_{cg}}\dot{\varphi}
\end{aligned}
\right\} \qquad (30)
$$

and (from equation (21))

$$
\ddot{\varphi} = \frac{\tan\psi}{\cos\theta}\ddot{\theta} + \frac{\dot{\theta}}{\cos\theta}\left[\frac{\dot{\psi}}{\cos^2\psi} + \dot{\theta}\tan\psi\tan\theta\right] \qquad (31)
$$

We are now in a position to formulate the projectile equations of motion. Letting $\overline{F}$ (which remains to be prescribed) denote the resultant applied force acting at the projectile c.g., and applying Newton's law of motion with $\overline{a}$ as prescribed in equation (25), there results the three scalar equations of translational motion of the projectile c.g. (written relative to S_o) in the form

$$
\left.
\begin{aligned}
F_{x_0} &= m_p\left[a_{x_0} + 2\omega_{y_0}\dot{z}_{cg} - 2\omega_{z_0}\dot{y}_{cg} + z_{cg}(\dot{\omega}_{y_0} + \omega_{x_0}\omega_{z_0}) - (y_{cg}+\varepsilon)(\dot{\omega}_{z_0} - \omega_{x_0}\omega_{y_0})\right] \\
F_{y_0} &= m_p\left[a_{y_0} + \ddot{y}_{cg} - 2\omega_{x_0}\dot{z}_{cg} - z_{cg}(\dot{\omega}_{x_0} - \omega_{z_0}\omega_{y_0}) - (y_{cg}+\varepsilon)(\omega_{x_0}^2 + \omega_{z_0}^2)\right] \\
F_{z_0} &= m_p\left[a_{z_0} + \ddot{z}_{cg} + 2\omega_{x_0}\dot{y}_{cg} + (y_{cg}+\varepsilon)(\dot{\omega}_{x_0} + \omega_{z_0}\omega_{y_0}) - z_{cg}(\omega_{x_0}^2 + \omega_{y_0}^2)\right]
\end{aligned}
\right\} \quad (32)
$$

where m_p denotes the mass of the projectile, and F_{x_o}, F_{y_o} and F_{z_o} denote the components of $\bar{F}$ along the respective axes of S_o.

Letting $\bar{M}$ (which also remains to be prescribed) denote the resultant applied moment acting about the projectile c.g., and applying the principle of angular momentum with $\bar{\omega}$ and $\dot{\bar{\omega}}$ prescribed respectively in equations (26) and (28), there results the three scalar equations which prescribe angular motion of the projectile (written relative to S in order to preclude introducing time derivatives of the projectile inertia tensor) in the form

$$
\left.
\begin{aligned}
M_x &= I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z + I_{xy}(\omega_z\omega_x - \dot{\omega}_y) - I_{xz}(\dot{\omega}_z + \omega_y\omega_x) - I_{yz}(\omega_y^2 - \omega_z^2) \\
M_y &= I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_z\omega_x + I_{yz}(\omega_x\omega_y - \dot{\omega}_z) - I_{yx}(\dot{\omega}_x + \omega_z\omega_y) - I_{zx}(\omega_z^2 - \omega_x^2) \\
M_z &= I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y + I_{zx}(\omega_y\omega_z - \dot{\omega}_x) - I_{zy}(\dot{\omega}_y + \omega_x\omega_z) - I_{xy}(\omega_x^2 - \omega_y^2)
\end{aligned}
\right\} \quad (33)
$$

where M_x, M_y and M_z denote the components of $\bar{M}$ about the respective axes of S, and I_{xx}, I_{yy}, ..., I_{zy} denote the elements of the projectile inertia tensor written relative to S.

Within the framework of the assumptions introduced, equations (32) and (33) present the general equations of projectile motion in terms of the applied loads and moments.

2.2 Simplification of Angular Velocity and Acceleration Expressions

The expressions for the terms representing the angular velocity and acceleration components entering equation (33) may be greatly simplified by noting that for most practical applications ψ and θ are sufficiently small such that

$$\left.\begin{aligned} \sin\psi \approx \psi \; ; \quad \cos\psi \approx 1 \\ \sin\theta \approx \theta \; ; \quad \cos\theta \approx 1 \end{aligned}\right\} \tag{34}$$

Under this condition it follows from equation (21) that

$$\dot{\varphi} \approx \psi\,\dot{\theta} \tag{35}$$

while from the mean value theorem it follows that the integral of of equation (35) will at most be of the order $\psi\theta$. Hence, to first order

$$\varphi \approx 0 \tag{36}$$

Imposing the above conditions and retaining only linear (first order) terms in ψ and θ, the direction cosines for the transformation from S to S_o, as given in equation (20), simplify to

$$\left.\begin{aligned} \ell_{11} = \ell_{22} = \ell_{33} = 1 \\ \ell_{12} = -\ell_{21} = -\psi \\ \ell_{13} = -\ell_{31} = \theta \\ \ell_{23} = \ell_{32} = 0 \end{aligned}\right\} \tag{37}$$

Noting that equations (18) and (30) simplify respectively to

$$\left.\begin{aligned} \omega_{x_{cg}} &\approx \psi\dot{\theta} - \theta\dot{\psi} \\ \omega_{y_{cg}} &\approx \dot{\theta} \\ \omega_{z_{cg}} &\approx \dot{\psi} \end{aligned}\right\} \tag{38}$$

and

$$\left.\begin{aligned} \dot{\omega}_{x_{cg}} &\approx \psi\ddot{\theta} - \theta\ddot{\psi} \\ \dot{\omega}_{y_{cg}} &\approx \ddot{\theta} + \psi\dot{\psi}\dot{\theta} \\ \dot{\omega}_{z_{cg}} &\approx \ddot{\psi} - \theta\dot{\theta}\dot{\psi} - \psi\dot{\theta}^2 \end{aligned}\right\} \tag{39}$$

and substituting these expressions, along with equation (37) into equations (27) and (29), there results the greatly simplified expressions for the angular velocity and acceleration components entering equation (33) in the form

$$\left.\begin{aligned} \omega_x &= \omega_{x_0} + \psi(\omega_{y_0} + \dot{\theta}) - \theta(\omega_{z_0} + \dot{\psi}) \\ \omega_y &= \omega_{y_0} + \dot{\theta} - \psi\,\omega_{x_0} \\ \omega_z &= \omega_{z_0} + \dot{\psi} + \theta\,\omega_{x_0} \end{aligned}\right\} \tag{40}$$

and

$$\left.\begin{aligned}
\dot{\omega}_x &= \dot{\omega}_{x_0} + \psi(\ddot{\theta} + \dot{\omega}_{y_0}) - \theta(\ddot{\psi} + \dot{\omega}_{z_0}) + \dot{\theta}(\theta\omega_{x_0} + \omega_{z_0}) + \dot{\psi}(\psi\omega_{x_0} - \omega_{y_0}) \\
\dot{\omega}_y &= \dot{\omega}_{y_0} + \ddot{\theta} + \psi(\dot{\theta}\dot{\psi} - \dot{\omega}_{x_0}) - \dot{\psi}(\omega_{x_0} + \psi\omega_{y_0}) + \psi\dot{\theta}(\theta\omega_{x_0} + \omega_{z_0}) \\
\dot{\omega}_z &= \dot{\omega}_{z_0} + \ddot{\psi} - \theta(\dot{\theta}\dot{\psi} - \dot{\omega}_{x_0}) - \psi\dot{\theta}^2 + \dot{\theta}(\omega_{x_0} - \theta\omega_{z_0}) + \theta\dot{\psi}(-\psi\omega_{x_0} + \omega_{y_0})
\end{aligned}\right\} \tag{41}$$

2.3 Applied Loads and Moments

The equations of projectile motion previously developed require specification of the applied loads, $\bar{F}$ (written relative to S_o), and the applied moments, $\bar{M}$ (written relative to S), for completion of the formulation. These loads and moments arise as a consequence of the projectile weight, interfacial contact of the rotating band and bourrelet with the bore, propellant gas pressure acting at the base of the projectile, and compressed air ahead of the projectile.

2.3.1 Projectile Weight Loading

The load applied to the projectile c.g. due to its own weight is given by

$$\bar{F}_w = -m_p g \hat{j}' \tag{42}$$

where m_p and g denote respectively the projectile mass and gravitational acceleration.

Applying the transformation given in equation (1), equation (42) is written in component form (relative to S_o) for application to equation (32), as follows

$$
\left.\begin{aligned}
F_{x_0}^{w} &= -m_p g \sin\psi_0 \cos\theta_0 \\
F_{y_0}^{w} &= -m_p g (\sin\psi_0 \sin\theta_0 \sin\varphi_0 + \cos\psi_0 \cos\varphi_0) \\
F_{z_0}^{w} &= -m_p g (\sin\psi_0 \sin\theta_0 \cos\varphi_0 - \cos\psi_0 \sin\varphi_0)
\end{aligned}\right\} \quad (43)
$$

2.3.2 Rotating-Band/Bore and Bourrelet/Bore Interfacial Contact Loadings

The loads and moments applied at the projectile c.g. due to interfacial contact of the rotating band and bourrelet with the bore are derived subject to the following assumptions:

(i) radial deformation of the rotating band and bourrelet are characterized by Winkler foundation models; rendering rotating-band/bore and bourrelet/bore interfacial load distributions which are directed radially, with local magnitude determined by resultant local radial displacement;

(ii) both the rotating band and bourrelet are permitted radial pre-loaded initial states;

(iii) the instantaneous annular segments defining rotating-band/bore and bourrelet/bore interfacial contact are kinematically determined by the extent of annular compression within the rotating band and bourrelet at each instant;

(iv) the projectile main-body is assumed to be rigid compared to the rotating band and bourrelet;

(v) curvature of the gun tube axis between the planes containing the rotating band and bourrelet is neglected;

(vi) rifling torque is transmitted without slippage via a uniformly distributed circumferential load acting over the instantaneous annular segment of the rotating band in contact with the bore;

(vii) the Euler angles ψ and θ satisfy equation (34).

In view of Assumptions (iv) and (vii), the displacement relative to S_o, $\bar{\delta}$, of any point of the projectile in a plane perpendicular to its geometric axis is given by

$$\bar{\delta} = (y_{cg} + \ell\psi)\hat{j}_o + (z_{cg} - \ell\theta)\hat{k}_o \tag{44}$$

where ℓ denotes the perpendicular distance from the projectile c.g. to the plane of interest. It is noted that $\ell > 0$ implies that the plane of interest is forward of the projectile c.g., $\ell = 0$ implies that the plane of interest contains the projectile c.g., while $\ell < 0$ implies that the plane of interest is aft of the c.g.

The radial component of this displacement, δ_r, is given as

$$\delta_r = \bar{\delta}\cdot\hat{i}_r = (y_{cg} + \ell\psi)\cos\varphi + (z_{cg} - \ell\theta)\sin\varphi \tag{45}$$

where φ denotes the angle between the projection of the y_o-axis onto the plane of interest and the line from the gun tube centerline to the point under consideration.

The maximum radial displacement, δ_{max}, in the plane of interest is determined by setting

$$\frac{\partial\delta_r}{\partial\varphi} = 0 \tag{46}$$

Letting φ_{max} denote the orientation of maximum radial displacement, there results from equation (45)

$$\varphi_{max} = \tan^{-1}\left(\frac{z_{cg} - \ell\theta}{y_{cg} + \ell\psi}\right) \tag{47}$$

and hence,

$$\delta_{max} = \sqrt{(y_{cg} + \ell\psi)^2 + (z_{cg} - \ell\theta)^2} \tag{48}$$

In view of equations (47) and (48), there results from equation (45)

$$\delta_r = \delta_{max}\cos(\varphi - \varphi_{max}) \tag{49}$$

Equation (49) denotes the radial component of displacement of any point in the plane of interest, and in particular, it may be used to specify the radial displacement of the planes containing the rotating band and bourrelet at the bore surface. It should be noted that the radial displacement as given by equation (49) is symmetric with respect to the angle φ_{max}.

In the sequel, quantities referred to the plane containing the bourrelet are assigned the subscript "1", quantities referred to the plane containing the rotating band are assigned the subscript "2". Furthermore, δ_{max} is replaced by either δ_1 or δ_2, and φ_{max} is replaced by either φ_1 or φ_2.

Assuming that the bourrelet is forward of the projectile c.g., while the rotating band is aft of the c.g., the radial displacement at the bourrelet due to projectile motion is given by

$$\delta_{r_1} = \delta_1\cos(\varphi - \varphi_1) \tag{50}$$

where,

$$\left.\begin{aligned} \delta_1 &= \sqrt{(y_{cg}+\ell_1\psi)^2+(z_{cg}-\ell_1\theta)^2} \\ \varphi_1 &= \tan^{-1}\left(\frac{z_{cg}-\ell_1\theta}{y_{cg}+\ell_1\psi}\right) \end{aligned}\right\} \tag{51}$$

while the radial displacement at the rotating band is given by

$$\delta_{r_2} = \delta_2 \cos(\varphi - \varphi_2) \tag{52}$$

where,

$$\left.\begin{aligned} \delta_2 &= \sqrt{(y_{cg}-\ell_2\psi)^2+(z_{cg}+\ell_2\theta)^2} \\ \varphi_2 &= \tan^{-1}\left(\frac{z_{cg}+\ell_2\theta}{y_{cg}-\ell_2\psi}\right) \end{aligned}\right\} \tag{53}$$

Letting δ_1^o and δ_2^o respectively denote the initial radial pre-compressions of the bourrelet and rotating band when the bore centerline and projectile geometric axis are initially aligned, it follows from Assumptions (i) thru (iv) that the radial loads per unit circumferential length are given as

$$R_i(\varphi) = k_i[\delta_i^o + \delta_i \cos(\varphi - \varphi_i)] \; ; \quad i = 1,2 \tag{54}$$

where k_i (i=1,2) respectively denote the radial spring stiffness per unit circumferential length of the bourrelet and rotating band.

It is noted that the expressions presented in equation (54) are valid for compression only, that is, for

$$[\delta_i^o + \delta_i \cos(\varphi - \varphi_i)] \geq 0 \; ; \quad i = 1,2 \tag{55}$$

Defining φ_{c_i} (i=1,2) as the instantaneous maximum annular range (measured from φ_i) for compressive loading of the bourrelet and rotating band, equation (54) may be written in the form

$$R_i(\varphi) = \begin{cases} k_i\left[\delta_i^o + \delta_i \cos(\varphi - \varphi_i)\right] \; ; & (\varphi_i + \varphi_{c_i}) \geq \varphi \geq (\varphi_i - \varphi_{c_i}) \\ 0 \; ; & (\varphi_i - \varphi_{c_i}) > \varphi > (\varphi_i + \varphi_{c_i}) \end{cases} \tag{56}$$

It follows from equation (55) and the definition of φ_{c_i} that when $\delta_i^o = 0$, $\varphi_{c_i} = \pi/2$. Hence, in the absence of pre-compression, loading takes place on half of the bore surface, and this loading is symmetric with respect to the angular orientation of maximum radial displacement. When $\delta_i^o \geq \delta_i$ it follows that $\varphi_{c_i} = \pi$. Hence, when displacement due to projectile motion is less than the pre-compression, contact is maintained over the full bore surface. For intermediate cases, namely $0 < \delta_i^o < \delta_i$, it follows that

$$\varphi_{c_i} = \tan^{-1}\left[-\sqrt{\left(\frac{\delta_i}{\delta_i^o}\right)^2 - 1}\,\right] \; ; \; i = 1,2 \tag{57}$$

The radial load distribution corresponding to an intermediate case is depicted in Figure 3.

The instantaneous interfacial contact loads acting on a differential element of either the rotating band or bourrelet are as depicted in Figure 4.

Under Assumption (vi), the friction force, $\mu\bar{R}$, acts tangential to the rifling surface in the direction opposing projectile motion at the surface. The angle α depicted in Figure 4 is given by

$$\alpha = \tan^{-1}(t_w r_b) \tag{58}$$

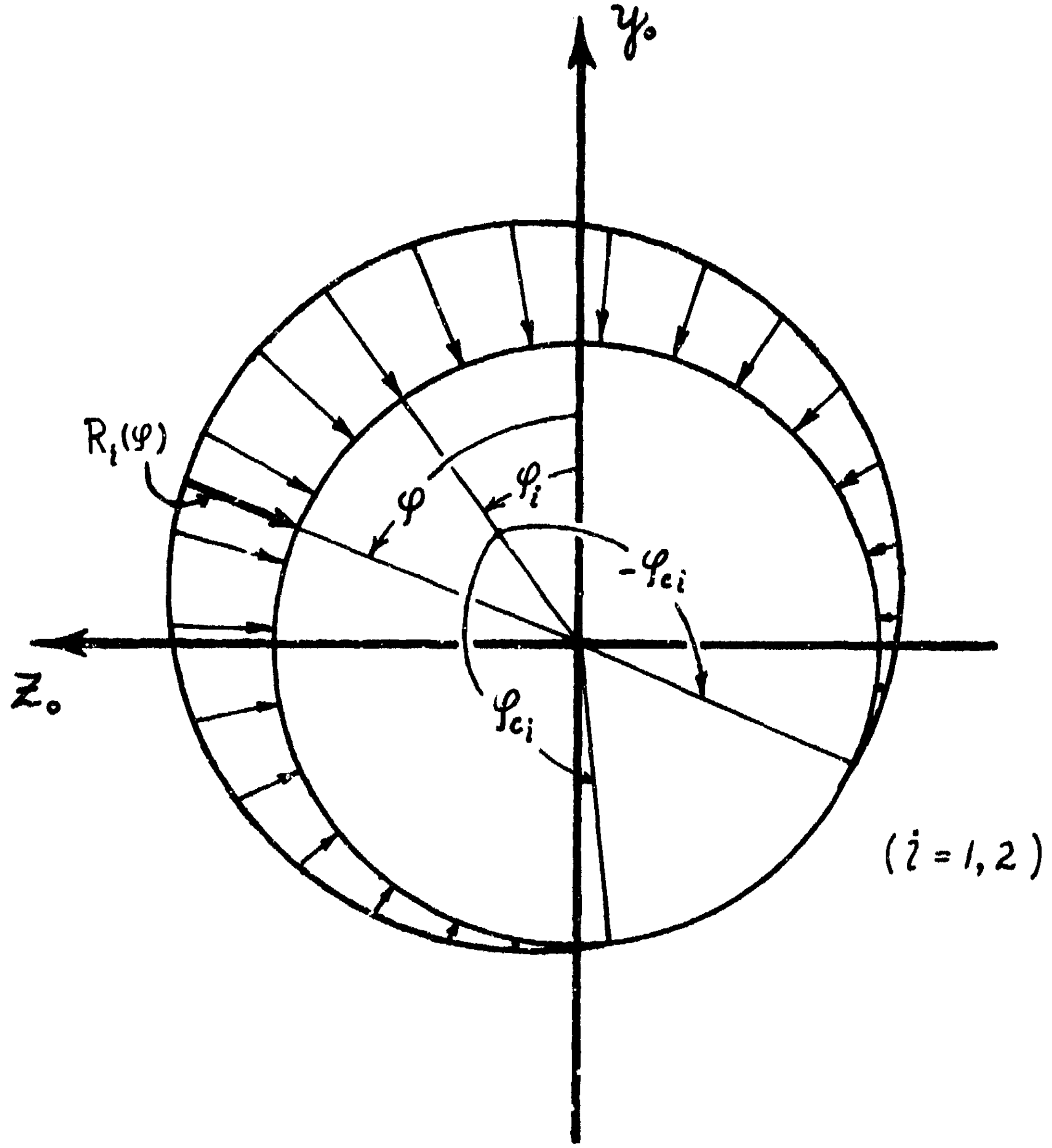

Figure 3 - Radial Load Distribution for Intermediate Case

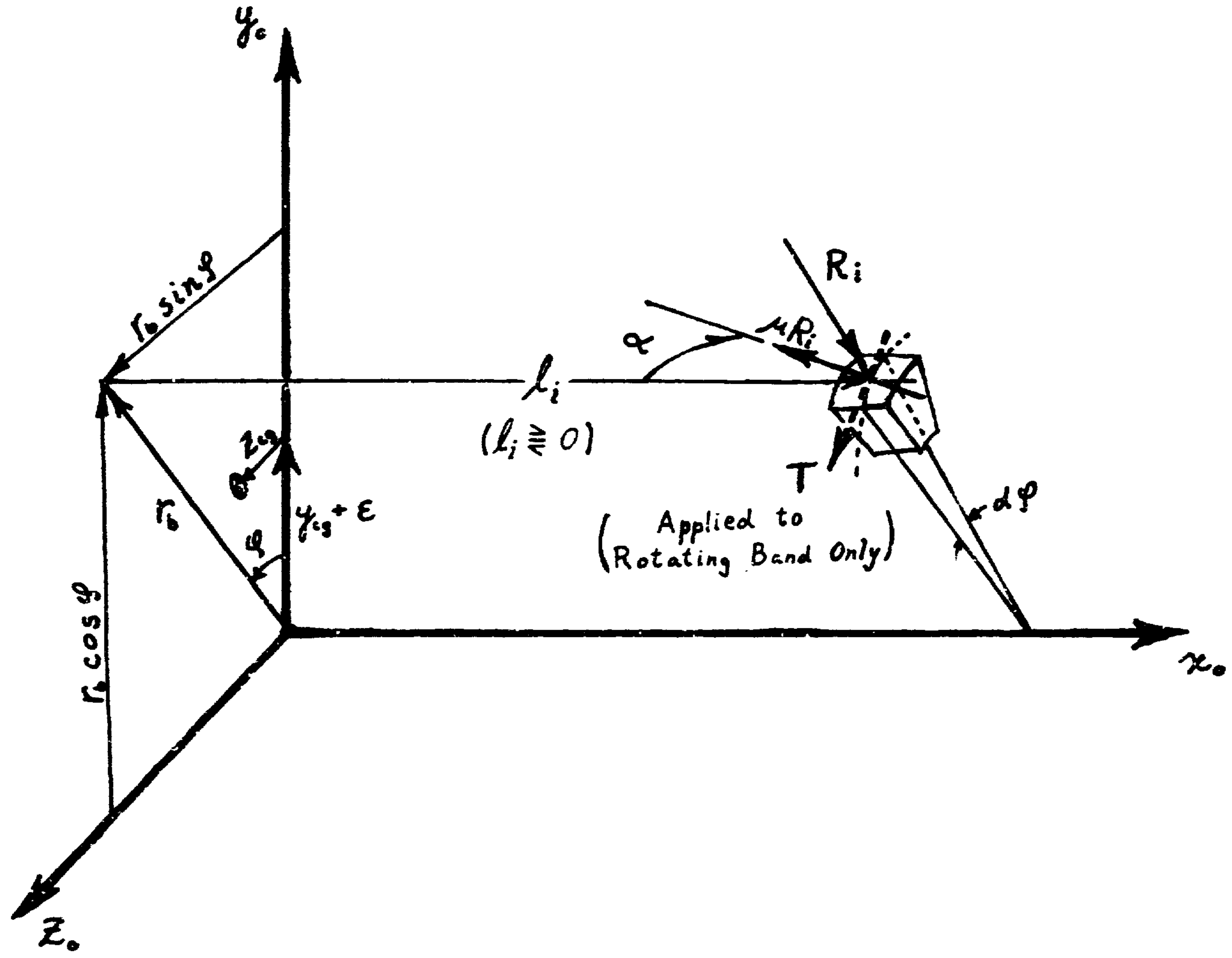

Figure 4 - <u>Loading on Typical Differential Element of either Bourrelet or Rotating Band</u>

where r_b and t_w denote respectively the bore radius and rifling twist. In addition, in accordance with Assumption (vi) the torque, T, applied to the rotating band per unit circumferential length is assumed to be a constant independent of φ over the range of compression.

Letting $d\bar{F}_i^c$ denote the incremental resultant contact load acting on a differential element of either the bourrelet or rotating band, it may be written in component form (relative to S_o) as

$$d\bar{F}_i^c = dF_{x_{o_i}}^c \hat{i}_o + dF_{y_{o_i}}^c \hat{j}_o + dF_{z_{o_i}}^c \hat{k}_o \qquad (59)$$

where referring to Figure 4, there results for the bourrelet

$$\left.\begin{aligned} dF_{x_{o_1}}^c &= -r_b \mu_1 R_1(\varphi) \cos\alpha \, d\varphi \\ dF_{y_{o_1}}^c &= r_b R_1(\varphi)[-\cos\varphi + \mu_1 \sin\alpha \sin\varphi] d\varphi \\ dF_{z_{o_1}}^c &= r_b R_1(\varphi)[-\sin\varphi - \mu_1 \sin\alpha \cos\varphi] d\varphi \end{aligned}\right\} \qquad (60)$$

and for the rotating band

$$\left.\begin{aligned} dF_{x_{o_2}}^c &= -r_b \mu_2 R_2(\varphi) \cos\alpha \, d\varphi \\ dF_{y_{o_2}}^c &= r_b \{R_2(\varphi)[-\cos\varphi + \mu_2 \sin\alpha \sin\varphi] - T\sin\varphi\} d\varphi \\ dF_{z_{o_2}}^c &= r_b \{R_2(\varphi)[-\sin\varphi - \mu_2 \sin\alpha \cos\varphi] + T\cos\varphi\} d\varphi \end{aligned}\right\} \qquad (61)$$

where μ_1 and μ_2 denote respectively the coefficients of friction at the bourrelet/bore and rotating-band/bore interfaces.

Hence, the resultant force applied to the projectile c.g. is obtained by integrating equation (59) over the respective rotating-band/bore and bourrelet/bore interfacial contact surfaces defined by $(\varphi_i - \varphi_{c_i}) \le \varphi \le (\varphi_i + \varphi_{c_i})$; i = 1, 2.

Substituting equation (54) into equations (60) and (61), and performing the required integrations for both the rotating band and bourrelet, there results the desired contributions to the force components acting at the projectile c.g. (relative to S_o) due to rotating band and bourrelet interfacial contact with the bore, namely

$$
\left.
\begin{aligned}
F_{x_o}^{c} &= -2r_b\cos\alpha\left[k_1\mu_1(\delta_1^{o}\varphi_{c_1} + \delta_1\sin\varphi_{c_1}) + k_2\mu_2(\delta_2^{o}\varphi_{c_2} + \delta_2\sin\varphi_{c_2})\right] \\
F_{y_o}^{c} &= r_b\{k_1[\mu_1\sin\alpha\sin\varphi_1 - \cos\varphi_1][2\delta_1^{o}\sin\varphi_{c_1} + \delta_1(\varphi_{c_1} + \sin\varphi_{c_1}\cos\varphi_{c_1})]+ \\
&\quad + k_2[\mu_2\sin\alpha\sin\varphi_2 - \cos\varphi_2][2\delta_2^{o}\sin\varphi_{c_2} + \delta_2(\varphi_{c_2} + \sin\varphi_{c_2}\cos\varphi_{c_2})]+ \\
&\quad - 2T\sin\varphi_2\sin\varphi_{c_2}\} \\
F_{z_o}^{c} &= r_b\{-k_1[\mu_1\sin\alpha\cos\varphi_1 + \sin\varphi_1][2\delta_1^{o}\sin\varphi_{c_1} + \delta_1(\varphi_{c_1} + \sin\varphi_{c_1}\cos\varphi_{c_1})]+ \\
&\quad - k_2[\mu_2\sin\alpha\cos\varphi_2 + \sin\varphi_2][2\delta_2^{o}\sin\varphi_{c_2} + \delta_2(\varphi_{c_2} + \sin\varphi_{c_2}\cos\varphi_{c_2})]+ \\
&\quad + 2T\cos\varphi_2\sin\varphi_{c_2}\}
\end{aligned}
\right\} \quad (62)
$$

Referring again to Figure 4, the moment arm, $\bar{l}_{M_i}$, from the projectile c.g. to the load acting on the differential element is given by

$$
\bar{l}_{M_i} = l_i\hat{i}_o + [r_b\cos\varphi - (y_{cg} + \varepsilon)]\hat{j}_o + [r_b\sin\varphi - z_{cg}]\hat{k}_o \quad (63)
$$

Hence, the resultant moment applied to the projectile c.g. is given by

$$\bar{M}^c = \sum_{i=1}^{2} \int_{\varphi_i - \varphi_{c_i}}^{\varphi_i + \varphi_{c_i}} \bar{\ell}_{M_i} \times d\bar{F}_i^c \tag{64}$$

Substituting equations (54), (60), (61) and (63) into equation (64), and performing the required integrations for both the rotating band and bourrelet, there results the contribution to the moment components acting at the projectile c.g. (relative to S_o) due to rotating band and bourrelet interfacial contact with the bore, namely

$$\left.\begin{aligned}
M_{x_o}^c &= r_b\left[\tan\alpha\, F_{x_o}^c - 2r_b T \varphi_{c_2}\right] - (y_{cg}+\varepsilon) F_{z_o}^c + z_{cg} F_{y_o}^c \\
M_{y_o}^c &= k_1 r_b\left[\ell_1(\mu_1 \sin\alpha\cos\varphi_1 + \sin\varphi_1) - r_b\mu_1\cos\alpha\sin\varphi_1\right]\left[2\delta_1^o \sin\varphi_{c_1} + \delta_1(\varphi_{c_1} + \sin\varphi_{c_1}\cos\varphi_{c_1})\right] + \\
&\quad - k_2 r_b\left[\ell_2(\mu_2 \sin\alpha\cos\varphi_2 + \sin\varphi_2) + r_b\mu_2\cos\alpha\sin\varphi_2\right]\left[2\delta_2^o \sin\varphi_{c_2} + \delta_2(\varphi_{c_2} + \sin\varphi_{c_2}\cos\varphi_{c_2})\right] + \\
&\quad + 2r_b \ell_2 T\cos\varphi_2 \sin\varphi_{c_2} - z_{cg}\tan\alpha\, F_{x_o}^c \\
M_{z_o}^c &= k_1 r_b\left[\ell_1(\mu_1 \sin\alpha\sin\varphi_1 - \cos\varphi_1) + r_b\mu_1\cos\alpha\cos\varphi_1\right]\left[2\delta_1^o \sin\varphi_{c_1} + \delta_1(\varphi_{c_1} + \sin\varphi_{c_1}\cos\varphi_{c_1})\right] + \\
&\quad + k_2 r_b\left[-\ell_2(\mu_2 \sin\alpha\sin\varphi_2 - \cos\varphi_2) + r_b\mu_2\cos\alpha\cos\varphi_2\right]\left[2\delta_2^o \sin\varphi_{c_2} + \delta_2(\varphi_{c_2} + \sin\varphi_{c_2}\cos\varphi_{c_2})\right] + \\
&\quad + 2r_b \ell_2 T\sin\varphi_2 \sin\varphi_{c_2} - (y_{cg}+\varepsilon)\tan\alpha\, F_{x_o}^c
\end{aligned}\right\} \tag{65}$$

The desired contribution to the moment components M_x, M_y and M_z (written relative to S), is obtained applying the transformation given in equation (20), from which there results noting Assumption (vii)

$$\left.\begin{aligned} M_x^c &= M_{x_0}^c + \psi M_{y_0}^c - \theta M_{z_0}^c \\ M_y^c &= M_{y_0}^c - \psi M_{x_0}^c \\ M_z^c &= M_{z_0}^c + \theta M_{x_0}^c \end{aligned}\right\} \tag{66}$$

2.3.3 Propellant Gas Pressure and Compressed Air Loadings

The loads and moments applied to the projectile c.g. due to propellant gas pressure acting at the base of the projectile and compressed ("ram") air ahead of the projectile are derived subject to the following assumptions:

(i) both the projectile base and "ram" air pressures are assumed to be known functions of time only, and at any instant are uniformly distributed over the respective projectile surfaces over which they act;

(ii) both the rotating band and bourrelet are assumed to act as ideal seals (i.e., no gas leakage);

(iii) the Euler angles ψ and θ satisfy equation (34).

Under Assumptions (i) and (ii), the base and "ram" air pressures act as effective hydrostatic pressures; generating a force equal to $- p \cdot \bar{n} \cdot dA$ on the differential surface area element, dA, with unit outward normal $\bar{n}$. Further, from general principles of hydrostatics, these pressure loadings may be replaced by resultant forces acting at the geometric center of, and directed perpendicular to, the planes containing the bourrelet and rotating band. Hence, the points of application of these resultant forces are defined by the respective intersection of each of these planes with the gun tube centerline, as shown in Figure 5.

Since $\hat{i}$ is parallel to the geometric axis of the projectile, it follows that

$$\left.\begin{aligned} \bar{F}_a &= -p_a A \hat{i} \\ \bar{F}_b &= p_b A \hat{i} \end{aligned}\right\} \tag{67}$$

where p_a and p_b denote respectively the instantaneous "ram" air and projectile base pressures, and A denotes the projected area of the bourrelet and rotating band planes perpendicular to $\hat{i}$.

Under Assumption (iii), it follows that

$$A \approx \pi r_b^2 \tag{68}$$

Hence, the resultant pressure load, $\bar{F}_p$, is given by

$$\bar{F}_p = (p_b - p_a)\pi r_b^2 \hat{i} \tag{69}$$

which, noting that in view of Assumption (iii)

$$\hat{i} = \hat{i}_o + \psi \hat{j}_o - \theta \hat{k}_o \tag{70}$$

may be written in component form (relative to S_o) for application to equation (32) as

$$\left.\begin{aligned} F_{x_o}^p &= (p_b - p_a)\pi r_b^2 \\ F_{y_o}^p &= (p_b - p_a)\pi r_b^2 \psi \\ F_{z_o}^p &= -(p_b - p_a)\pi r_b^2 \theta \end{aligned}\right\} \tag{71}$$

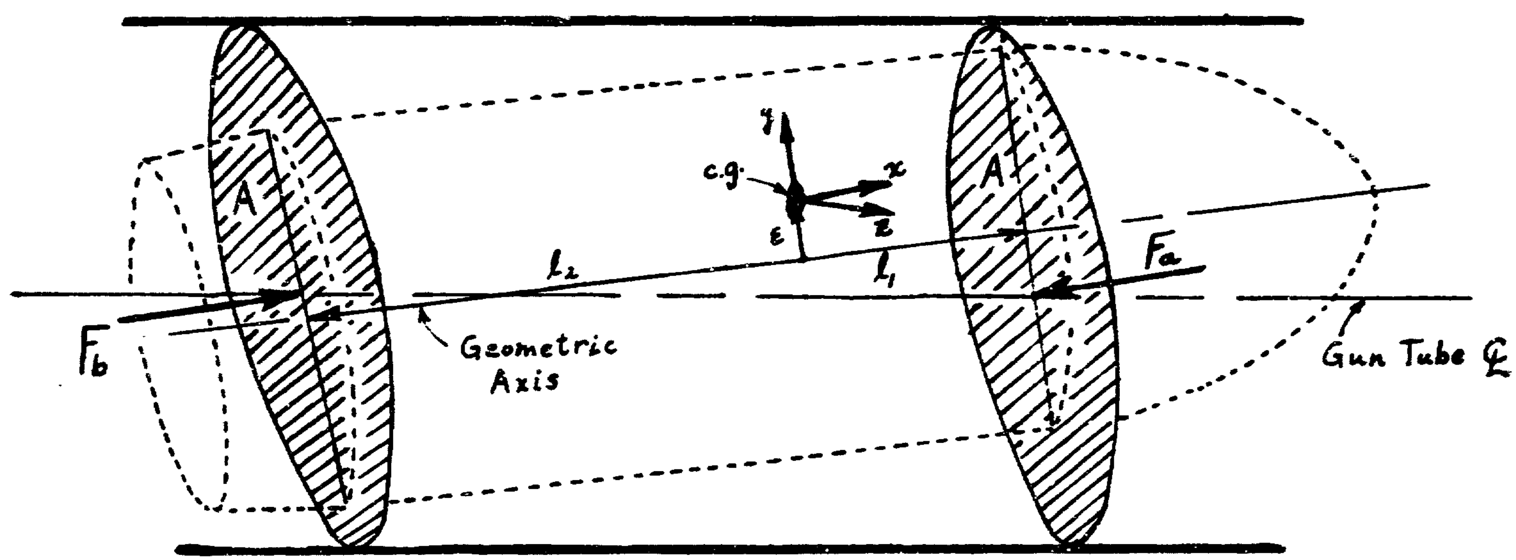

Figure 5 - Projectile Base and "Ram" Air Pressure Loads

Once again in view of Assumption (iii), it may be shown that the moment arm, $\bar{\ell}_a$, from the projectile c.g. to the "ram" air load, $\bar{F}_a$, is given by

$$\bar{\ell}_a = \ell_1 \hat{i} - (y_{cg} + \varepsilon + \ell_1 \psi)\hat{j} - (z_{cg} - \ell_1 \theta)\hat{k} \tag{72}$$

while the corresponding moment arm, $\bar{\ell}_b$, to $\bar{F}_b$ is

$$\bar{\ell}_b = -\ell_2 \hat{i} - (y_{cg} + \varepsilon - \ell_2 \psi)\hat{j} - (z_{cg} + \ell_2 \theta)\hat{k} \tag{73}$$

Hence, the resultant moment applied to the projectile c.g. is

$$\bar{M}_p = \bar{\ell}_a \times \bar{F}_a + \bar{\ell}_b \times \bar{F}_b \tag{74}$$

Substituting equations (67), (72) and (73) into equation (74), and performing the indicated vector operations, there results the desired contribution to the moment components acting at the projectile c.g. (relative to S) due to propellant gas and "ram" air pressures, in the form

$$\left.\begin{aligned} M_x^p &= 0 \\ M_y^p &= -\pi r_b^2 [(p_b - p_a) z_{cg} + \theta(p_b \ell_2 + p_a \ell_1)] \\ M_z^p &= \pi r_b^2 [(p_b - p_a)(y_{cg} + \varepsilon) - \psi(p_b \ell_2 + p_a \ell_1)] \end{aligned}\right\} \tag{75}$$

2.3.4 Summation of Applied Loads and Moments

Summing corresponding force components from equations (43), (62) and (71), there results for application to equation (32)

$$\left.\begin{aligned} F_{x_o} &= F_{x_o}^{w} + F_{x_o}^{c} + F_{x_o}^{p} \\ F_{y_o} &= F_{y_o}^{w} + F_{y_o}^{c} + F_{y_o}^{p} \\ F_{z_o} &= F_{z_o}^{w} + F_{z_o}^{c} + F_{z_o}^{p} \end{aligned}\right\} \qquad (76)$$

Similarly, summing corresponding moment components from equations (66) and (75), there results for application to equation (33)

$$\left.\begin{aligned} M_x &= M_x^{c} \\ M_y &= M_y^{c} + M_y^{p} \\ M_z &= M_z^{c} + M_z^{p} \end{aligned}\right\} \qquad (77)$$

2.4 Solution Technique

Within the framework of the assumptions introduced, equations (32) and (33), with the applied loads and moments as defined in equations (76) and (77), prescribe projectile in-bore motion in terms of projectile design data, interior ballistics data, and gun tube design and motion data. Of these required data, gun tube motion data are not known a priori, and hence, equations (32) and (33) must be solved simultaneously with the equations of the gun dynamics simulation code (References 1 and 2).

Equations (32), (33), (76) and (77) consist of a system of six simultaneous ordinary differential equations for the determination of the six unknowns v_p, y_{cg}, z_{cg}, ψ, θ, and T. However, noting that v_p is to be prescribed by interior ballistics data, the first equation in (32) may be deleted (or retained for use as a check); rendering a system of five simultaneous equations for incorporation within the gun dynamics simulation code.

To incorporate the projectile motion equations within the simulation code, it is noted that the parameters representing gun tube motion in equations (32) and (33), namely, the components of $\tilde{R}_o$, their time derivatives, the Euler angles, ψ_o, θ_o and φ_o, and their time derivatives, are related to gun tube motion parameters defined in the simulation code. In addition, the negative of the interfacial contact loads between the rotating-band and bore, and bourrelet and bore, as applied to the projectile, must be applied to the gun tube. The specific details of incorporating the projectile in-bore motion equations herein developed within the gun dynamics simulation code will be presented in detail in a separate report.

3. COMPARISON WITH OTHER PROJECTILE DESCRIPTIONS AND FORMULATIONS

The formulation herein developed is general in that it prescribes the full six degree-of-freedom motion of a projectile of finite geometry and inertia traveling in a flexible gun tube. The degrees-of-freedom selected correspond to three orthogonal translational motions of the projectile c.g. relative to the gun tube axis and three (Eulerian) rotations of the projectile about its c.g. (related to projectile pitch, yaw and roll motions). Since gun tube motion (which is unrestricted within the formulation) is not known a priori, the projectile equations of motion herein formulated will be solved simultaneously with the equations of the gun dynamics simulation code previously developed (References 1 and 2).

A similar, but far more restrictive formulation has recently been presented by S.H. Chu (Reference 3). Chu permits the projectile three degrees-of-freedom; two orthogonal translational motions of the projectile c.g. relative to the gun tube and one rotational (pitching) motion. As a consequence of neglecting the remaining degrees-of-freedom, the gun tube centerline, the projectile c.g., and the resultants of the rotating band and bourrelet contact-load distributions with the bore all lie in the same plane. Hence, Chu's formulation is essentially planar; whereas, the formulation herein presented is of general three-dimensional character.

Another recent, but also restrictive projectile motion formulation has been presented by H.L. Langhaar and A.P. Boresi (Reference 4). Langhaar and Boresi present a rigorous kinematical description of a point moving along a time-dependent space curve.

The point is identified with the geometric center of a rigid projectile. The time-dependent space curve is identified with the centerline of a flexible gun tube. The projectile is further characterized such that its geometric center and c.g. coincide (which precludes the ability to investigate the effects of mass eccentricity), and such that its geometric axis is directed along the instantaneous tangent to the gun tube centerline (which precludes the ability to investigate the effects of projectile pitch and yaw motions relative to the gun tube). The projectile is permitted two degrees-of-freedom relative to the gun tube; translational motion of its c.g. along the gun tube centerline and rotational motion about the centerline (corresponding to projectile spin). Accounting for rotary inertia of the projectile about its spin axis, there results a traveling point-mass projectile load with superposed gyroscopic couple.

J.J. Wu (Reference 5) also adopts a traveling point-mass projectile description, but with superposed traveling pitching moment due to mass eccentricty (while neglecting rotary inertia about the pitch axis). Several other investigators (References 6 thru 8) have adopted the simpler traveling point-mass description, with and without mass eccentricity and projectile spin (while neglecting rotary inertia about the spin axis).

4. CONCLUSIONS

A system of simultaneous differential equations has been formulated which describes the general motion of a projectile of finite geometry and inertia traveling in a flexible gun tube. The formulation permits the projectile six degrees-of-freedom relative to the gun tube; three orthogonal translational motions of the projectile c.g. relative to the gun tube axis and three (Eulerian) rotational motions of the projectile about its c.g. (related to projectile pitch, yaw and roll motions). The formulation is presented in terms of interior ballistics data, projectile design data, and gun tube design and motion data. The formulation accounts for projectile spin, mass eccentricity, elastic deformation of the projectile rotating band and bourrelet, and projectile/bore interfacial friction and torque transmission. Furthermore, the formulation is unrestricted regarding the nature of gun tube motion.

The formulation herein contained has been compared with projectile descriptions and formulations employed by other investigators and, based upon this comparison, is considered to be the most generally applicable formulation appearing in recent literature.

Since gun tube motion is not known a priori, and indeed since projectile motion affects gun tube motion and vice versa, the projectile in-bore motion equations herein formulated will be incorporated within and solved simultaneously with the gun dynamics simulation code equations previously developed; permitting replacement of the traveling point-mass projectile description presently contained within the code. Incorporation of the projectile motion equations within the simulation code and simultaneous solution with the gun dynamics equations, as well as comparison of the results obtained with previously obtained results treating the projectile as a traveling point-mass, will be presented in a separate report.

5. REFERENCES

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APPENDIX

The instantaneous angular orientation of the coordinate system S_o, with coordinates (x_o, y_o, z_o) and unit triad $(\hat{i}_o, \hat{j}_o, \hat{k}_o)$, relative to the coordinate system S', with coordinates (x', y', z') and unit triad $(\hat{i}', \hat{j}', \hat{k}')$, is defined in terms of the Euler angles $(\psi_o, \theta_o, \varphi_o)$ depicted in Figure A.

Referring to Figure A, the instantaneous angular orientation of S_o is achieved by subjecting S' to the following consecutive rotations:

(i) ψ_o about z', bringing x' to its final elevation, ξ, and y' to its intermediate orientation, η;

(ii) θ_o about η, bringing ξ to its final azimuth, x_o, and z' to its final azimuth, ζ;

(iii) φ_o about x_o, bringing η to its final orientation, y_o, and ζ to its final elevation, z_o.

The direction cosines, ℓ^o_{ij}, defining the transformation between S_o and S' are obtained by noting the relation between the unit triads depicted in Figure A subsequent to each consecutive rotation, as follows:

(i) following the rotation ψ_o, there results

$$\hat{i}' = \cos\psi_o\, \hat{i}_\xi - \sin\psi_o\, \hat{i}_\eta$$
$$\hat{j}' = \sin\psi_o\, \hat{i}_\xi + \cos\psi_o\, \hat{i}_\eta$$

(ii) following the rotation θ_o, there results

$$\hat{i}_\xi = \cos\theta_o\, \hat{i}_o + \sin\theta_o\, \hat{i}_\zeta$$
$$\hat{k}' = -\sin\theta_o\, \hat{i}_o + \cos\theta_o\, \hat{i}_\zeta$$

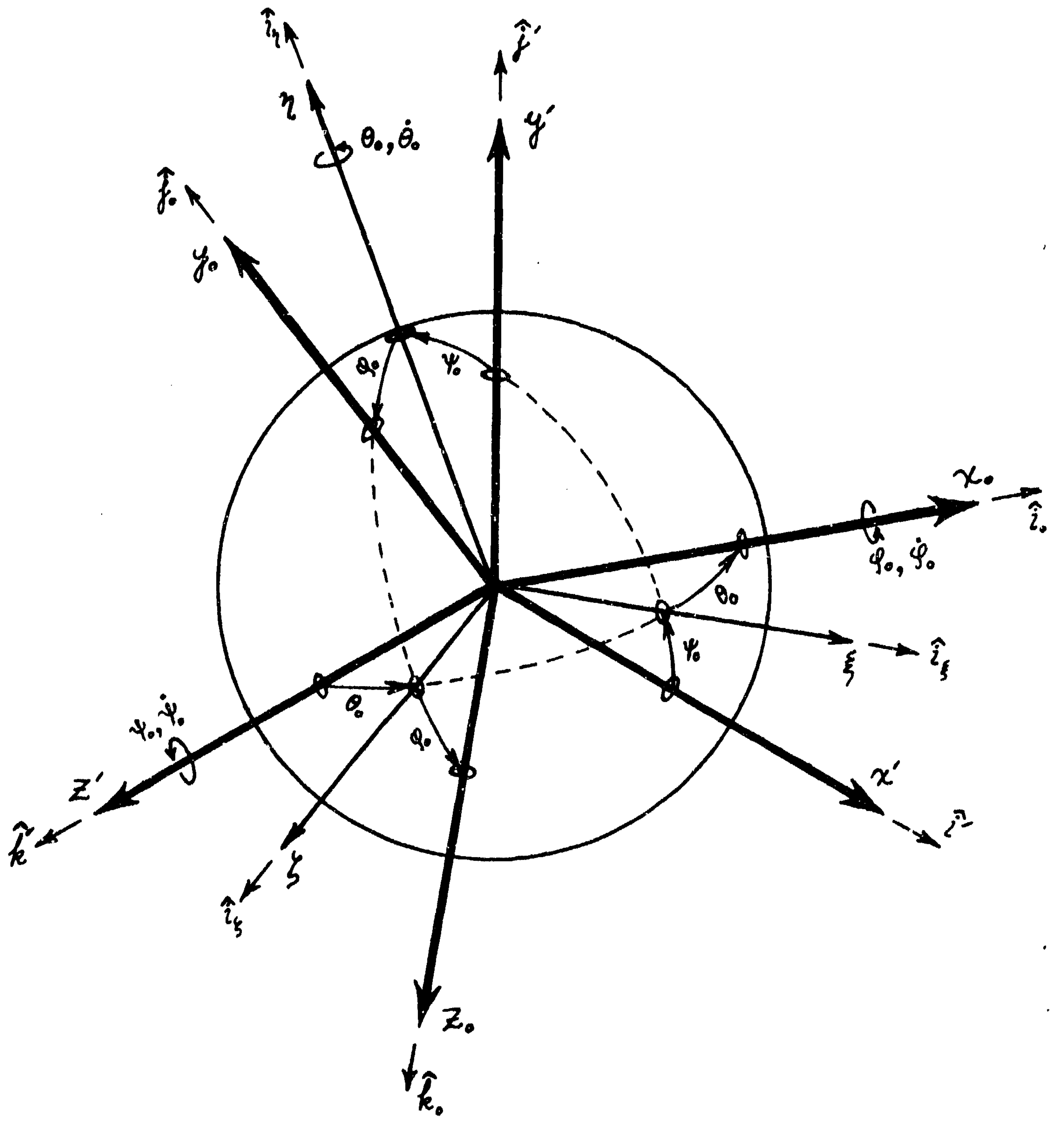

Figure A - Definition of Euler Angles

(iii) following the rotation φ_o, there results

$$\hat{i}_\eta = \cos\varphi_o \, \hat{j}_o - \sin\varphi_o \, \hat{k}_o$$

$$\hat{i}_\zeta = \sin\varphi_o \, \hat{j}_o + \cos\varphi_o \, \hat{k}_o$$

Eliminating the unit triad $(\hat{i}_\xi, \hat{i}_\eta, \hat{i}_\zeta)$ associated with the intermediate coordinates (ξ, η, ζ), there results the transformation presented in equation (1), with the direction cosines as defined in equation (2).

The relation between the angular velocity of S_o relative to S', namely $\bar{\omega}_o$, and the Euler angles, is obtained by noting that

$$\bar{\omega}_o = \dot{\psi}_o \hat{k}' + \dot{\theta}_o \hat{i}_\eta + \dot{\varphi}_o \hat{i}_o$$

and substituting the above transformations between unit triads to obtain $\bar{\omega}_o$ in the form prescribed in equation (11); which renders the components as given in equation (12).